

ON THE FLAG f -VECTOR OF A GRADED LATTICE WITH NONTRIVIAL HOMOLOGY

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ABSTRACT. It is proved that the Boolean algebra of rank n minimizes the flag f -vector among all graded lattices of rank n , whose proper part has nontrivial top-dimensional homology. The analogous statement for the flag h -vector is conjectured in the Cohen-Macaulay case.

1. INTRODUCTION

Let P be a finite graded poset of rank $n \geq 1$, having a minimum element $\hat{0}$, maximum element $\hat{1}$ and rank function $\rho : P \rightarrow \mathbb{N}$ (we refer to [12, Chapter 3] for any undefined terminology on partially ordered sets). Given $S \subseteq [n-1] := \{1, 2, \dots, n-1\}$, the number of chains $\mathcal{C} \subseteq P \setminus \{\hat{0}, \hat{1}\}$ such that $\{\rho(x) : x \in \mathcal{C}\} = S$ will be denoted by $f_P(S)$. For instance, $f_P(S)$ is equal to the number of elements of P of rank k , if $S = \{k\} \subseteq [n-1]$, and to the number of maximal chains of P , if $S = [n-1]$. The function which maps S to $f_P(S)$ for every $S \subseteq [n-1]$ is an important enumerative invariant of P , known as the *flag f -vector*; see, for instance, [4].

The present note is partly motivated by the results of [2, 6]. There it is proven that the Boolean algebra of rank n minimizes the **cd**-index, an invariant which refines the flag f -vector, among all face lattices of convex polytopes and, more generally, Gorenstein* lattices, of rank n . It is natural to consider lattices which are not necessarily Eulerian, in this context. To state our main result, we fix some more notation as follows. We denote by $\Delta(Q)$ the simplicial complex consisting of all chains in a finite poset Q , known as the *order complex* [5] of Q , and by $\tilde{H}_*(\Delta; \mathbf{k})$ the reduced simplicial homology over \mathbf{k} of an abstract simplicial complex Δ , where \mathbf{k} is a fixed field or \mathbb{Z} . We denote by B_n the Boolean algebra of rank n (meaning, the lattice of subsets of the set $[n]$, partially ordered by inclusion) and recall that if $S = \{s_1 < s_2 < \dots < s_l\} \subseteq [n-1]$, then $f_{B_n}(S)$ is equal to the multinomial coefficient $\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, \dots, n - s_l}$.

Theorem 1.1. *Let L be a finite graded lattice of rank n , with minimum element $\hat{0}$ and maximum element $\hat{1}$, and let $\bar{L} = L \setminus \{\hat{0}, \hat{1}\}$ be the proper part of L . If $\tilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0$, then*

$$(1.1) \quad f_L(S) \geq \alpha_n(S)$$

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for every $S \subseteq [n-1]$. In other words, the Boolean algebra of rank n minimizes the flag f -vector among all finite graded lattices of rank n whose proper part has nontrivial top-dimensional reduced homology over \mathbf{k} .

A similar statement, asserting that the Boolean algebra of rank n has the smallest number of elements among all finite lattices L satisfying $\tilde{H}_{n-2}(\Delta(\bar{L}); \mathbb{Z}) \neq 0$, was proved by Meshulam [10]. The proof of Theorem 1.1, given in Section 2, is elementary and similar in spirit to (but somewhat more involved than) the proof of the result of [10]. A different (but less elementary) proof may be given using the methods of [6, Section 2]. In the remainder of this section we discuss some consequences of Theorem 1.1 and a related open problem.

The f -vector of a simplicial complex Δ is defined as the sequence $f(\Delta) = (f_0, f_1, \dots)$, where f_i is the number of i -dimensional faces of Δ . We recall that the order complex $\Delta(\bar{B}_n)$ is isomorphic to the barycentric subdivision of the $(n-1)$ -dimensional simplex. The next statement follows from this observation, Theorem 1.1 and the fact (see, for instance, [13, p. 95]) that each entry of the f -vector of the order complex $\Delta(\bar{L})$ can be expressed as a sum of entries of the flag f -vector of L .

Corollary 1.2. *The barycentric subdivision of the $(n-1)$ -dimensional simplex has the smallest possible f -vector among all order complexes of the form $\Delta(\bar{L})$, where L is a finite graded lattice of rank n satisfying $\tilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0$.*

Analogous results for the class of flag simplicial complexes have appeared in [1, 7, 9, 11].

Let P be a graded poset of rank n , as in the beginning of this section. The flag h -vector of P is the function assigning to each $S \subseteq [n-1]$ the integer

$$(1.2) \quad h_P(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_P(T).$$

Equivalently, we have

$$(1.3) \quad f_P(S) = \sum_{T \subseteq S} h_P(T)$$

for every $S \subseteq [n-1]$. We write $\beta_n(S)$ for the entry $h_{B_n}(S)$ of the flag h -vector of the Boolean algebra of rank n and recall [12, Corollary 3.12.2] that $\beta_n(S)$ is equal to the number of permutations of $[n]$ with descent set S .

It is known that if P is Cohen-Macaulay over \mathbf{k} (see [5, Section 11] or [12, Section 3.8] for the definition), then $h_P(S) \geq 0$ for every $S \subseteq [n-1]$. Moreover, in this case $\Delta(\bar{L})$ has nontrivial top-dimensional reduced homology over \mathbf{k} if and only if $\mu_P(\hat{0}, \hat{1}) \neq 0$, where μ_P is the Möbius function of P . Hence, Theorem 1.1 implies that the Boolean algebra of rank n minimizes the flag f -vector among all Cohen-Macaulay lattices of rank n with nonzero Möbius number. In view of (1.3), the following conjecture provides a natural strengthening of this statement.

Conjecture 1.3. *Let L be a finite lattice of rank n , with minimum element $\hat{0}$ and maximum element $\hat{1}$. If L is Cohen-Macaulay over \mathbf{k} and $\mu_L(\hat{0}, \hat{1}) \neq 0$, then*

$$(1.4) \quad h_L(S) \geq \beta_n(S)$$

for every $S \subseteq [n - 1]$. In other words, the Boolean algebra of rank n minimizes the flag h -vector among all Cohen-Macaulay lattices of rank n with nonzero Möbius number.

This conjecture was initially stated by the author under the assumption that $\mu_L(x, y) \neq 0$ holds for all $x, y \in L$ with $x \leq_L y$ and took its present form after a question raised by R. Stanley [14], asking whether this condition could be relaxed to $\mu_L(\hat{0}, \hat{1}) \neq 0$. It would imply that among all Cohen-Macaulay order complexes of the form $\Delta(\bar{L})$, where L is a lattice of rank n satisfying $\mu_L(\hat{0}, \hat{1}) \neq 0$, the barycentric subdivision of the $(n - 1)$ -dimensional simplex has the smallest possible h -vector (the entries of the h -vector of this subdivision are the Eulerian numbers, counting permutations of the set $[n]$ by the number of descents). Conjecture 1.3 is known to hold for Gorenstein* lattices (in this case it follows from the stronger result [6, Corollary 1.3], mentioned earlier, on the \mathbf{cd} -index of such a lattice) and for geometric lattices [3, Proposition 7.4].

2. PROOF OF THEOREM 1.1

Throughout this section, L is a lattice as in Theorem 1.1. For $a, b \in L$ with $a \leq_L b$, we denote by $\Delta(a, b)$ (respectively, by $\Delta(a, b]$) the order complex of the open interval (a, b) (respectively, half-open interval $(a, b]$) in L . We say that an element $x \in L$ is *good* if $x = \hat{0}$ or $\tilde{H}_{k-2}(\Delta(\hat{0}, x); \mathbf{k}) \neq 0$, where k is the rank of x in L , and otherwise that x is *bad*.

The proof of Theorem 1.1 will follow from the next proposition.

Proposition 2.1. *Under the assumptions of Theorem 1.1, the lattice L has at least $\binom{n}{k}$ good elements of rank k for every $k \in \{0, 1, \dots, n\}$.*

Proof. We proceed in several steps.

Step 1: We show that L has at least one good coatom. Suppose, by the way of contradiction, that no such coatom exists. Suppose further that L has the minimum possible number of coatoms among all lattices of rank n which satisfy the assumptions of Theorem 1.1 and have no good coatom. Since $\Delta(\bar{L})$ is non-acyclic over \mathbf{k} , the order complex $\Delta(\bar{L})$ cannot be a cone and hence L must have at least two coatoms. Let c be one of them and consider the complexes $\Delta(\bar{L} \setminus \{c\})$ and $\Delta(\hat{0}, c]$. The union of these complexes is equal to $\Delta(\bar{L})$ and their intersection is equal to $\Delta(\hat{0}, c)$. Since $\Delta(\hat{0}, c]$ is a cone, hence contractible, and since $\tilde{H}_{n-3}(\Delta(\hat{0}, c); \mathbf{k}) = 0$ by assumption, it follows from the Mayer-Vietoris long exact sequence in homology for $\Delta(\bar{L} \setminus \{c\})$ and $\Delta(\hat{0}, c]$ that

$$(2.1) \quad \tilde{H}_{n-2}(\Delta(\bar{L} \setminus \{c\}); \mathbf{k}) \cong \tilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0.$$

Since $L \setminus \{c\}$ may not be graded, we consider the subposet $M = J \cup \{\hat{1}\}$ of L , where J stands for the order ideal of L generated by all coatoms other than c . The poset M is a finite meet-semilattice with a maximum element and hence it is a lattice by [12, Proposition

3.3.1]. Since L is graded of rank n , so is M and the set of $(n-1)$ -element chains of $\Delta(\bar{M})$ coincides with that of $\Delta(\bar{L} \setminus \{c\})$, where $\bar{M} = M \setminus \{\hat{0}, \hat{1}\}$ is the proper part of M . The last statement and (2.1) imply that

$$\tilde{H}_{n-2}(\Delta(\bar{M}); \mathbf{k}) \cong \tilde{H}_{n-2}(\Delta(\bar{L} \setminus \{c\}); \mathbf{k}) \neq 0.$$

Clearly, all coatoms of M are bad. Since M has one coatom less than L , we have arrived at the desired contradiction.

Step 2: Assume that $n \geq 2$ and let b be any coatom of L . We show that there exists an atom a of L which is not comparable to b and satisfies $\tilde{H}_{n-3}(\Delta(a, \hat{1}); \mathbf{k}) \neq 0$. Arguing by contradiction, once again, suppose that no such atom exists. Suppose further that the number of atoms of L which do not belong to the interval $[\hat{0}, b]$ is as small as possible for a graded lattice L of rank n and coatom b which have this property and satisfy the assumptions of Theorem 1.1. Since $\Delta(\bar{L})$ is non-acyclic over \mathbf{k} , the Crosscut Theorem of Rota [5, Theorem 10.8] implies that there exists at least one atom of L which does not belong to the interval $[\hat{0}, b]$. Let a be any such atom and let M be the subposet of L consisting of $\hat{0}$ and the elements of the dual order ideal of L generated by the atoms other than a . The arguments in Step 1, applied to the dual of L , show that M is a graded lattice of rank n which satisfies $\tilde{H}_{n-2}(\Delta(\bar{M}); \mathbf{k}) \cong \tilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0$. Since $M \setminus (\hat{0}, b]$ has one atom less than $L \setminus (\hat{0}, b]$, this contradicts our assumptions on L and b .

Step 3: We now show that L has at least n good coatoms by induction on n . The statement is trivial for $n = 1$, so suppose that $n \geq 2$. By replacing $L \setminus \{\hat{1}\}$ with its order ideal generated by the good coatoms, as in Step 1, we may assume that all coatoms of L are good. Let b be any coatom of L . By Step 2, there exists an atom a of L which is not comparable to b and satisfies $\tilde{H}_{n-3}(\Delta(a, \hat{1}); \mathbf{k}) \neq 0$. The interval $[a, \hat{1}]$ in L is a graded lattice of rank $n-1$ to which the induction hypothesis applies. Therefore, it has at least $n-1$ coatoms and all of these are different from b . It follows that L has at least n coatoms, all of which are good.

Step 4: We prove the following: Given any integers $0 \leq r \leq k \leq n$ and any order ideal I of $L \setminus \{\hat{1}\}$ generated by at most r elements, there exist at least $\binom{n-r}{k-r}$ good elements of L of rank k which do not belong to I . The special case $r = 0$ of this statement, in which I is the empty ideal, is equivalent to the proposition. Thus, it suffices to prove the statement.

We proceed by induction on n and $n-r$, in this order. The statement is trivial for $n = 1$ and for $r = n$, so we assume that $n \geq 2$ and $0 \leq r \leq n-1$. Consider an order ideal I of $L \setminus \{\hat{1}\}$ generated by at most r elements and let k be an integer in the range $r \leq k \leq n$. Since I contains at most $r \leq n-1$ coatoms of L , Step 3 implies that there exists a good coatom, say b , of L which does not belong to I . The interval $[\hat{0}, b]$ of L is a graded lattice of rank $n-1$ whose proper part has nontrivial top-dimensional reduced homology over \mathbf{k} . Moreover, the intersection $I \cap [\hat{0}, b]$ is an order ideal of $[\hat{0}, b]$ which is generated by at most r elements, namely the meets of b with the maximal elements of I . By our induction on n , there exist at least $\binom{n-r-1}{k-r}$ good elements of $[\hat{0}, b]$ of rank k which do not belong to I . The union $J = I \cup [\hat{0}, b]$ is an order ideal of $L \setminus \{\hat{1}\}$ which is generated

by at most $r + 1$ elements. By our induction on $n - r$, there exist at least $\binom{n-r-1}{k-r-1}$ good elements of L of rank k which do not belong to J . We conclude that there exist at least $\binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1} = \binom{n-r}{k-r}$ good elements of L of rank k which do not belong to I . This completes the inductive step and the proof of the statement. \square

Proof of Theorem 1.1. We proceed by induction on n . The result is trivial for $n = 1$ and for $S = \emptyset$, so we assume that $n \geq 2$ and choose a nonempty subset S of $[n - 1]$. We denote by k the largest element of S and observe that $f_L(S)$ is equal to the number of pairs (x, \mathcal{C}) , where x is an element of L of rank k and \mathcal{C} is a chain in the interval $[\hat{0}, x]$, such that the set of ranks of the elements of \mathcal{C} is equal to $S \setminus \{k\}$. By Proposition 2.1, there are at least $\binom{n}{k}$ good elements x of rank k in L and each of the intervals $[\hat{0}, x]$ is a graded lattice of rank k whose proper part has nontrivial top-dimensional reduced homology over \mathbf{k} . Thus, the induction hypothesis applies to these intervals and we may conclude that

$$f_L(S) \geq \binom{n}{k} \alpha_k(S \setminus \{k\}) = \alpha_n(S).$$

This completes the induction and the proof of the theorem. \square

We end with a note on the case of equality in (1.1). It was shown in [10] that every lattice L which satisfies $\tilde{H}_{n-2}(\Delta(\bar{L}); \mathbb{Z}) \neq 0$ and has cardinality 2^n must be isomorphic to the Boolean algebra B_n . As a result, if equality holds in (1.1) for every singleton $S \subseteq [n - 1]$, then L is isomorphic to B_n . Using the arguments in this section, as well as induction on n and k , the following statement has been verified by Kolins and Klee [8]: if L satisfies the assumptions of Theorem 1.1 and for some $k \in \{1, 2, \dots, n - 1\}$ equality holds in (1.1) for every subset S of $[n - 1]$ of cardinality k , then L is isomorphic to the Boolean algebra of rank n .

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